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A NOTE ON BICOMPLEX MANIFOLD

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Abstract: Bicomplex number can be regarded as a well-known extension of complex number and is of dimension four. In this paper, we define a new notion of manifolds termed as almost bicomplex, bicomplex, bicomplex Hermite manifold and discuss some interesting properties of Nijenhuis tensor, contravariant almost analytic vector fields etc. in this sequel.

Keywords and Phrases: Complex manifold, Bicomplex manifold, almost bicomplex manifold, bicomplex Hermite manifold.

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1. Introduction, Definitions and Notations

Bicomplex numbers, a well-known extension of complex numbers of dimension four have been studied for quite a long time and a lot of work has been done in this area. In 1844 the skew field of quaternions was introduced by W. R. Hamilton, a well-known extension of the field of complex numbers {cf. [8]}. In quaternions, there are three imaginary units i, j, k that anti-commute with the property $ij = k$.

The beauty of the theory of quaternions is that they form a field where all the ordinary operations can be accomplished. Although from the algebraic point of view the lack of commutativity is not such a terrible problem. But this lack of commutativity opens a new direction of thought where it is considered as a four dimensional algebra containing $\mathbb{C}$ as a subalgebra that preserves the commutativity property. This can be done by considering two imaginary units i, j with $ij = ji = k$. This k is known as a hyperbolic imaginary unit, i.e., an element such that $k^2 = 1$.

In 1848, J. Cockle wrote a series of papers in which he introduced a new algebra that he called the algebra of tessarines {cf. [3], [4], [5] & [6]}. Inspired by the work of Hamilton and Cockle, in 1892, Carrado Segre published a paper in which he defined an infinite set of algebras and gave the concept of bicomplex numbers. In this paper, he gave us the idempotent elements $\frac{1+ij}{2}$ and $\frac{1-ij}{2}$ which plays most important role in the entire theory of bicomplex analysis. After that, a few other mathematicians namely Spampinato and S. Dragoni developed the first rudiments of function theory on bicomplex numbers.

The next major push in the study of bicomplex analysis was given by J. D. Riley [13], in 1953 he developed the theory of functions of bicomplex variables. But the most recommendable contribution was done by G. B. Price [12], the theory of holomorphic functions of a bicomplex variable as well as multicomplex variables is widely developed in his work.

Definition 1.1. [11] *The set of bicomplex numbers* $\mathbb{C}_2$ *is defined by* $\mathbb{C}_2 = \{z : z = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ *or equivalently* $\mathbb{C}_2 = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C}_1\}$, *where* $\mathbb{C}_1$ *is the set of complex numbers with imaginary unit* i *such that* $i^2 = j^2 = -k^2 = -1$ *and* $ij = ji = k$.

Definition 1.2. [11] *For any bicomplex number* $z = z_1 + z_2\mathbf{j}$ *the conjugation is defined in the following way:*

$$\overline{z}_i = \overline{z_1} + \overline{z_2}\mathbf{j}, \quad \overline{z}_j = z_1 - z_2\mathbf{j}, \quad \overline{z}_k = \overline{z_1} - \overline{z_2}\mathbf{j}.$$

Definition 1.3. [7] *A complex structure on a finite dimensional real vector space* V *is an endomorphism* F *(i.e.,* $F : V \to V$ *is a vector valued real linear function) such that*

$$F(F(X)) = -X \quad, \forall X \in V$$
$$i.e., \quad F^2 = -I,$$

where I *stands for identity transformation.*
A real vector space V *with complex structure* F *can be turned into a complex vector space by defining scalar multiplication by complex number as follows:*

$$(a + ib)X = aX + bF(X),$$

for $X \in V$ *and* $a, b, c, d \in \mathbb{R}$. *Clearly the real dimension* m *of* V *must be even and* $m/2$ *is the complex dimension of* V.

Definition 1.4. [7] *An almost complex structure on a real differentiable manifold* M *of dimension* $n(n = 2m, m$ *is a positive integer) is a tensor field* F *which is at every point* x *of* M, *an endomorphism of the tangent space* $T_x(M)$ *such that* $F^2 = -I$, *where* I *denotes the identity transformation* $T_x(M)$. *A manifold with a fixed almost complex structure is called an almost complex manifold.*

Definition 1.5. [7] *A vector field is said to be contravariant almost analytic if*

$$(L_V F)(X) = 0,$$

where L_V *denotes the Lie derivative with respect to the vector field* V.

Definition 1.6. [7] *A vector field is said to be strictly contravariant almost analytic if*

$$\begin{aligned} (L_V F)(X) &= 0, \quad \textit{and} \\ (L_{F(V)} F)(X) &= 0, \end{aligned}$$

where L_V *denotes the Lie derivative with respect to the vector field* V.

Definition 1.7. [7] *An almost complex manifold endowed with a metric* g *such that*

$$g(F(X), F(Y)) = g(X, Y)$$

is called an almost Hermite manifold and (F, g) *is called an almost Hermite structure.*

Before proving our main results, we have defined the following definitions which are required to prove our results.

Definition 1.8. *Let* $F_1, F_2 : V \to V$ *be vector valued functions such that*

$$\begin{aligned} & F_1(F_1(X)) = -X \quad , \quad F_2(F_2(X)) = -X \\ & \textit{and} \\ & F_1(F_2(X)) = F_2(F_1(X) = X, \quad \forall X \in V \\ \textit{i.e.,} \quad & F_1^2 = -I \quad , \quad F_2^2 = -I, \quad \textit{and} \quad F_1F_2 = F_2F_1 = I, \end{aligned}$$

where I *stands for identity transformation.*

A real vector space V *with complex structures* F_1 *and* F_2 *can be turned into a bicomplex vector space by defining scalar multiplication by bicomplex number as follows:*

$$(a + i_1 b + i_2 c + i_1 i_2 d)X = aX + bF_1(X) + cF_2(X) + dF_1(F_2(X)),$$

for all $X \in V$ *and* $a, b, c, d \in \mathbb{R}$.

Definition 1.9. *An almost bicomplex structure on a real differentiable manifold* M *of dimension* n $(n = 4m, m \in N)$ *is a tensor field consists of* F_1 *and* F_2 *which is at every point* x *of* M*, an endomorphism of the tangent space* $T_x(M)$ *such that* $F_1^2 = -I$, $F_2^2 = -I$ *and* $F_1 F_2 = F_2 F_1$ *but* $F_1 \neq F_2$*, where* I *denotes the identity transformation* $T_x(M)$*. A manifold with a fixed almost bicomplex structure is called an almost bicomplex manifold.*

Definition 1.10. (Bicomplex manifold) *A bicomplex manifold is a Riemannian manifold endowed with integrable almost complex structures* F_1 *and* F_2 *with respect to the Riemannian metric and satisfy the relations* $F_1^2 = F_2^2 = -I$*. If* (M, g, F_1, F_2) *is a bicomplex manifold, then the tangent space* $T_x M$ *is a bicomplex vector space for each point* x *of* M*.*

Definition 1.11. *Let* F_1 *and* F_2 *be two almost complex structures in a bicomplex manifold* $M^n (n = 2m$*, where* m *is a positive integer). Nijenhuis tensor with respect to* F_1 *and* F_2 *is a vector valued bilinear function* N *defined by*

$$N(X, Y) = [F_1(X), F_2(Y)] + [X, Y] - F_1[F_2(X), Y] - F_2[X, F_1(Y)]$$

where $X, Y \in \chi(M)$ *and* $[,]$ *stands for Lie bracket.*

Definition 1.12. *An almost bicomplex Hermite manifold endowed with a metric* g *such that*

$$g(F_1(X), F_2(Y)) = g(X, Y),$$

where $g(F_1(X), F_1(Y)) = -g(X, Y)$ *and* $g(F_2(X), F_2(Y)) = -g(X, Y)$ *is called an almost bicomplex Hermite manifold and* (F_1, F_2, g) *is called an almost bicomplex Hermite structure.*

Definition 1.13. *A vector field is said to be contravariant almost bicomplex analytic if*

$$(L_V F_1)(X) = 0 \quad \text{and} \quad (L_V F_2)(X) = 0,$$

where L_V *denotes the Lie derivative with respect to the vector field* V*.*

Definition 1.14. *A vector field* X *is said to be strictly contravariant almost bicomplex analytic if*

$$(L_V F_1)(X) = 0, \qquad (L_{F_1(V)} F_1)(X) = 0,$$

and

$$(L_V F_2)(X) = 0, \qquad (L_{F_2(V)} F_2)(X) = 0,$$

where L_V denotes the Lie derivative with respect to the vector field V.

Definition 1.15. *Let M^n be bicomplex manifold with bicomplex structures F_1 and F_2. The fundamental 2-form $\tilde{F_1}$ and $\hat{F_2}$ and M^n is defined by*

$$\tilde{F_1}(X,Y) = g(F_1(X),Y) \text{ and } \hat{F_2}(X,Y) = g(X,F_2(Y))$$

for all vector fields X and Y on M^n.

2. Lemmas

In this section, we present the following lemmas which will be needed in the sequel.

Lemma 2.1. *In an almost complex manifold $M^n(n = 2m)$, the almost complex structure F has m eigen values i and m eigen values $-i$.*

Lemma 2.2. *In an almost complex manifold a vector field X is contravariant almost analytic if and only if*

$$L_V F(X) = F(L_V X), \qquad or F(L_V F(X)) + L_V X = 0.$$

Lemma 2.3. *If a vector field V in an almost complex manifold M is strictly contravariant almost analytic, then $N(V,Y) = 0$ for every vector field X.*

3. Main results

In this section, we present the main results of the paper.

Theorem 3.1. *Let (M, F_1, F_2, g) be an almost bicomplex Hermite manifold. Then $g(F_1(X), F_2(Y)) = g(F_2(X), F_1(Y))$.*
Proof. Here

$$\begin{aligned} g(F_1(X), F_2(Y)) &= g(-F_2(X), F_2(Y)) \\ &= g(-F_2(X), -F_1(Y)) \\ &= g(F_2(X), F_1(Y)). \end{aligned}$$

Therefore, $g(F_1(X), F_2(Y)) = g(F_2(X), F_1(Y))$.

Theorem 3.2. *In an almost bicomplex manifold (M, F_1, F_2, g)*

(a) $N(X, F_1(X)) = F_1[F_1(X), F_2(X)]$

(b) $N(F_2(X), X) = F_2[F_1(X), F_2(X)]$

(c) $N(F_1(X), X) = -N(X, F_2(X)) = [F_1(X), X] + [F_2(X), X]$

(d) $N(F_1(X), Y) = [F_1(X), Y] - [X, F_2(Y)] - F_1[X, Y] - F_2[F_1(X), F_1(Y)]$

(e) $N(X, F_1(Y)) = [F_1(X), Y] + [X, F_1(Y)] - F_1[F_2(X), F_1(Y)] - F_2[X, Y]$

(f) $N(F_1(X), F_2(Y)) = [X, Y] + [F_1(X), F_2(Y)] - F_1[X, F_2(Y)] - F_2[F_1(X), Y]$

(g) $F_1(N(X, Y)) = F_1[F_1(X), F_2(Y)] + F_1[X, Y] + [F_2(X), Y] - [X, F_1(Y)]$

(h) $F_2(N(X, Y)) = F_2[F_1(X), F_2(Y)] + F_2[X, Y] - [F_2(X), Y] + [X, F_1(Y)]$

(i) $F_2(N(F_1(X), F_2(Y))) = F_2[X, Y] + F_2[F_1(X), F_2(Y)] - [X, F_2(Y)] + [F_1(X), Y]$

Proof. (a): We have

$$\begin{aligned} N(X, F_1(X)) &= [F_1(X), F_2F_1(X)] + [X, F_1(X)] - F_1[F_2(X), F_1(X)] \\ &\qquad - F_2[X, F_1^2(X)] \\ &= [F_1(X), X] + [X, F_1(X)] - F_1[F_2(X), F_1(X)] - F_2[X, X] \\ &= F_1[F_1(X), F_2(X)] \end{aligned}$$

This proves (a).
(b): We have

$$\begin{aligned} N(F_2(X), X) &= [F_1F_2(X), F_2(X)] + [F_2(X), X] - F_1[F_2^2(X), X] \\ &\qquad - F_2[F_2(X), F_1(X)] \\ &= [X, F_2(X)] + [F_2(X), X] + F_1[X, X] - F_2[F_2(X), F_1(X)] \\ &= F_2[F_1(X), F_2(X)] \end{aligned}$$

This proves (b).
(c):

$$\begin{aligned} N(X, F_2(X)) &= [F_1(X), F_2^2(X)] + [X, F_2(X)] - F_1[F_2(X), F_2(X)] \\ &\qquad - F_2[X, F_1F_2(X)] \\ &= -[F_1(X), X] + [X, F_2(X)] - F_1[F_2(X), F_2(X)] + F_2[X, X] \\ &= [X, F_1(X)] + [X, F_2(X)] \end{aligned}$$

and

$$\begin{aligned}
N(F_1(X), X) &= [F_1^2(X), F_2(X)] + [F_1(X), X] - F_1[F_2F_1(X), X] \\
&\qquad - F_2[F_1X, F_1(X)] \\
&= -[X, F_2(X)] + [F_1(X), X] - F_1[F_2F_1(X), X] \\
&\qquad - F_2[F_1(X), F_1(X)] \\
&= -[X, F_1(X)] - [X, F_2(X)] \\
&= -N(X, F_2(X))
\end{aligned}$$

This proves (c).
(d):

$$\begin{aligned}
N(F_1(X), Y) &= [F_1^2(X), F_2(Y)] + [F_1(X), Y] - F_1[F_2F_1(X), Y] \\
&\qquad - F_2[F_1(X), F_1(Y)] \\
&= [F_1(X), Y] - [X, F_2(Y)] - F_1[X, Y] - F_2[F_1(X), F_1(Y)]
\end{aligned}$$

This proves (d).
(e):

$$\begin{aligned}
N(X, F_1(Y)) &= [F_1(X), F_2F_1(Y)] + [X, F_1(Y)] - F_1[F_2(X), F_1(Y)] \\
&\qquad - F_2[X, F_1^2(Y)] \\
&= [F_1(X), Y] + [X, F_1(Y)] - F_1[F_2(X), F_1(Y)] - F_2[X, Y]
\end{aligned}$$

This proves (e).
(f):

$$\begin{aligned}
N(F_1(X), F_2(Y)) &= [F_1^2(X), F_2^2(Y)] + [F_1(X), F_2(Y)] \\
&\qquad - F_1[F_2F_1(X), F_2(Y)] - F_2[F_1(X), F_1F_2(Y)] \\
&= [X, Y] + [F_1(X), F_2(Y)] - F_1[X, F_2(Y)] - F_2[F_1(X), Y]
\end{aligned}$$

This proves (f).
(g):

$$\begin{aligned}
F_1(N(X, Y)) &= F_1[F_1(X), F_2(Y)] + F_1[X, Y] - F_1^2[F_2(X), Y] \\
&\qquad - F_1F_2[X, F_1(Y)] \\
&= F_1[F_1(X), F_2(Y)] + F_1[X, Y] + [F_2(X), Y] - [X, F_1(Y)]
\end{aligned}$$

This proves (g).
(h) :

$$\begin{aligned}F_2(N(X,Y)) &= F_2[F_1(X),F_2(Y)] + F_2[X,Y] - F_1F_1[F_2(X),Y] \\ &\qquad - F_2^2[X,F_1(Y)] \\ &= F_2[F_1(X),F_2(Y)] + F_2[X,Y] - [F_2(X),Y] + [X,F_1(Y)]\end{aligned}$$

This proves (h).
(i):

$$\begin{aligned}F_2(N(F_1(X),F_2(Y))) &= F_2[F_1F_1(X),F_2F_2(Y)] + F_2[F_1(X),F_2(Y)] \\ &\qquad - F_2F_1[F_2F_1(X),F_2(Y)] - F_2^2[F_1(X),F_1F_2(Y)] \\ &= F_2[X,Y] + F_2[F_1(X),F_2(Y)] - [X,F_2(Y)] \\ &\qquad + [F_1(X),Y]\end{aligned}$$

This completes the proof of (i).

Theorem 3.3. *In an almost bicomplex manifold $M^n(n=2m)$, the complex structures F_1 has $m/2$ eigen values i and $m/2$ eigen values $-i$ and F_2 has $m/2$ eigen values j and $m/2$ eigen values $-j$.*

Proof. Let $M^n(n=2m)$ be an almost bicomplex manifold with complex structures F_1 and F_2. That is, F_1 and F_2 are vector valued real linear functions on M such that

$$F_1(X) = \overline{X},\ F_2(X) = \overline{X} \text{ and } F_1^2 = -X,\ F_2^2 = -X \text{ but } F_1 \neq F_2.$$

Also, let ρ_1 and ρ_2 be eigen values of F_1 and F_2 corresponding to the eigen vectors Z and W respectively. Then we get that

$$F_1(Z) = \rho_1(Z) \text{ and } F_2(W) = \rho_2(W).$$

Therefore, $-Z = F_1^2(Z) = \rho_1(F(Z)) = \rho_1^2(Z), i.e., (\rho_1^2+1)Z = 0$ and $-W = F_2^2(W) = \rho_2(F(W)) = \rho_2^2(W), i.e., (\rho_2^2+1)W = 0$. Hence $\rho_1^2 = -1$ and $\rho_2^2 = -1$.
Since F_1 and F_2 are real valued linear functions and of rank $2m$, therefore there are $m/2$ pairs of eigen values i and $-i$ of F_1 and $m/2$ pairs of eigen values j and $-j$ of F_2.
This completes the proof.

Theorem 3.4. *In an almost bicomplex manifold $M^n(n=4m)$, a vector field X is contravariant almost bicomplex analytic if and only if*

$$\begin{aligned}&L_VF_1(X) = F_1(L_VX), &\text{or}\quad &F_1(L_VF_1(X)) + L_VX = 0,\\ &L_VF_2(X) = 0 & &F_2(L_VF_2(X)) + L_VX = 0.\end{aligned}$$

Proof. We have

$$(L_V F_1)(X) = L_V F_1(X) - F_1(L_V X)$$
$$and\ (L_V F_1)(X) = L_V F_1(X) - F_1(L_V X)$$

If V is contravariant almost bicomplex analytic, then

$$(L_V F_1)(X) = 0 \text{ and } (L_V F_2)(X) = 0.$$

So

$$(L_V F_1)(X) = F_1(L_V X)$$
$$or,\ \ F_1(L_V F_1(X)) = F_1^2(L_V X) = -L_V X$$

and

$$(L_V F_2) = F_2(L_V X)$$
$$or,\ \ F_2(L_V F_2(X)) = F_2^2(L_V X) = -L_V X,$$
$$i.e.,\ \ F_1(L_V F_1(X)) + L_V X = 0 \text{ and } F_2(L_V F_2(X)) + L_V X = 0.$$

Conversely, let $L_V F_1(X) = F_1(L_V X)$ and $L_V F_2(X) = F_2(L_V X)$. Then

$$(L_V F_1)(X) = L_V F_1(X) - F_1(L_V X) = 0$$
$$\text{and } (L_V F_2)(X) = L_V F_2(X) - F_2(L_V X) = 0$$

So, V is contravariant almost bicomplex analytic.

Theorem 3.5. *If a vector field V in an almost bicomplex manifold is strictly contravariant almost bicomplex analytic, then $N(V, Y) = 0$ for every vector field X.*

Proof. Let V be strictly contravariant almost bicomplex analytic.
Then

$$(L_V F_1)(X) = 0,\ \ (L_V F_2)(X) = 0$$
$$\text{and } (L_{F_1(V)} F)(X) = 0,\ \ (L_{F_2(V)} F)(X) = 0$$

From $L_V F_1(X) = F_1(L_V X)$ and $L_V F_2(X) = F_2(L_V X)$, we get that

$$[V, F_1(X)] = F_1([V, X]), \qquad [V, F_2(X)] = F_2([V, X])$$
$$\text{and } [F_1(V), F_1(X)] = F_1([F_1(V), X]), \qquad [F_2(V), F_2(X)] = F_2([F_2(V), X])$$

Now

$$\begin{aligned}N(V,X) =&[F_1(V),F_2(X)]+[V,X]-F_1[F_2(V),X]-F_2[V,F_1(X)]\\ =&F_2[F_1(V),(X)]+[V,X]-F_1[F_2(V),X]-F_2F_1[V,X]\\ =&-F_2[X,F_1(V)]+[V,X]+F_1[X,F_2(V)]-[V,X]\\ =&-F_2F_1[X,V]+F_1F_2[X,V]\\ =&-[X,V]+[X,V]=0.\end{aligned}$$

This proves the theorem.

Theorem 3.6. *In an almost bicomplex Hermite manifold the following relations hold:*

(a) $\tilde{F}_1(X,Y)=-\hat{F}_2(X,Y)$

(b) $\tilde{F}_1(F_1(X),F_2(Y))=-\hat{F}_2(X,Y)$

(c) $\tilde{F}_1(F_2(X),F_1(Y))=-\tilde{F}_1(X,Y)$

(d) $\hat{F}_2(F_1(X),F_2(Y))=-\tilde{F}_1(X,Y)$

(e) $\hat{F}_2(F_2(X),F_1(Y))=-\hat{F}_2(X,Y)$

Proof. (a): We know that $g(F_1(X),F_2(Y))=g(X,Y)$
Putting $X=F_1(X)$, we get

$$\begin{aligned}g(F_1^2(X),F_2(Y))&=g(F_1(X),Y)\\ or,\ -g(X,F_2(Y))&=g(F_1(X),Y)\\ or,\ -\hat{F}_2(X,Y))&=\tilde{F}_1(X,Y)\\ or,\ \tilde{F}_1(X,Y)&=-\hat{F}_2(X,Y)\end{aligned}$$

This completes the proof of (a).
(b): Putting $X=F_1(X)$ and $Y=F_2(Y)$ in $\tilde{F}_1(X,Y)=g(F_1(X),Y)$, we get

$$\begin{aligned}\tilde{F}_1(F_1(X),F_2(Y))&=g(-X,F_2(Y))\\ &=-g(X,F_2(Y))\\ &=-\hat{F}_2(X,Y)\end{aligned}$$

Therefore, $\tilde{F}_1(F_1((X), F_2(Y)) = -\hat{F}_2(X, Y)$
This completes the proof of (b).
(c): Putting $X = F_2(X)$ and $Y = F_1(Y)$ in $\tilde{F}_1(X, Y) = g(F_1(X), Y)$

$$\begin{aligned}\tilde{F}_1(F_2(X), F_1(Y)) &= g(F_1 F_2(X), F_1(Y)) \\ &= g(X, F_1(Y)) \\ &= -g(F_1(X), Y) \\ &= -\tilde{F}_1(X, Y)\end{aligned}$$

Therefore, $\tilde{F}_1(F_2(X), F_1(Y)) = -\tilde{F}_1(X, Y)$
This completes the proof of (c).
(d): Putting $X = F_1(X)$ and $Y = F_2(Y)$ in $\hat{F}_2(X, Y) = g(X, F_2(Y))$

$$\begin{aligned}\hat{F}_2(F_1(X), F_2(Y)) &= g(F_1(X), F_2^2(Y)) \\ &= -g(F_1(X), Y) \\ &= -\tilde{F}_1(X, Y)\end{aligned}$$

Therefore, $\hat{F}_2(F_1(X), F_2(Y)) = -\tilde{F}_1(X, Y)$.
This completes the proof of (d).
(e): Putting $X = F_2(X)$ and $Y = F_1(Y)$ in $\hat{F}_2(X, Y) = g(X, F_2(Y))$

$$\begin{aligned}\hat{F}_2(F_2(X), F_1(Y)) &= g(F_2(X), F_2 F_1(Y)) \\ &= g(F_2(X), Y) \\ &= -g(X, F_2(Y)) \\ &= -\hat{F}_2(X, Y)\end{aligned}$$

Therefore, $\hat{F}_2(F_2(X), F_1(Y)) = -\hat{F}_2(X, Y)$.
This completes the proof of (e).

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